On Wold's approach to representation of preferences☆

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ABSTRACT

This paper presents easily verifiable sufficient conditions on sequence spaces that guarantee representation of preference orders. Our approach involves identifying a suitable subset of the set of alternatives, such that (a) the preference order is representable on this subset, and (b) the subset has the property that for each alternative, there is some element in this subset which is indifferent to it. We follow Wold in choosing this subset to be the diagonal. Our first result uses a weak monotonicity condition (on the diagonal), and a substitution condition, and may be identified as the essence of Wold's contribution. In the second result, we show that one can obtain a Wold-type representation result when weak monotonicity is replaced by a weak continuity condition. We use the countable order-dense characterization of representability in the proofs of both results, thereby integrating the contributions of Wold (1943) and Debreu (1954). Through a series of examples we show that our representation results are robust; they cannot be improved upon by dropping any of our conditions. An example is also presented to show that existence of degenerate indifference classes is compatible with the representation of monotone preferences. Our study thereby indicates that while the presence of substitution possibilities can be useful in representing preferences, they are not necessary for such results to hold.

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1. Introduction

There are two principal methods leading to the representability of a preference order by a numerical utility function. The first is due to Wold (1943) who proposed that if preferences were monotone, and if for every consumption bundle, there was a unique diagonal bundle to which it was indifferent, then the scalar associated with that diagonal bundle can be used as a numerical measure of the utility of the consumption bundle. The second is due to Debreu (1954), who showed that if the set of consumption bundles contained a *countable order-dense* subset, then the preference order can be represented by a numerical utility function.

The condition of Debreu turns out to be *necessary* as well for the numerical representation of a preference order [see Fishburn (1970), Kreps (1988) and Bridges and Mehta (1995)]. On the other hand, following Debreu (1954), non-representability

crucially builds on the lexicographic preferences[1] he introduced. The inability to represent lexicographic preferences may be viewed as arising from the fact that there are no substitution possibilities, since each consumption bundle is indifferent only to itself.

The emphasis on substitution possibilities for the representation of preference orders appear in the writing of Georgescu-Roegen (1954). In his study on consumer preferences, Chipman (1960, p. 210) says that the countable order dense property "has little intuitive appeal". He suggests an Axiom of Substitution as part of his formal axiomatic set up. The Wold approach can be seen as identifying a specific form of substitution which is sufficient to guarantee representability of monotone preferences (see Beardon and Mehta (1994) for an exposition of the Wold approach).

The substitution condition used by Wold is not necessary, even for the class of monotone preferences. Nevertheless, because of its transparent geometric intuition, the method of Wold has been widely used to establish numerical representation of preference orders under a variety of different assumptions on preferences [see, for example, Diamond (1965), Asheim et al. (2012), Mitra and Ozbek (2013) and Banerjee (2014)].[2]

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[1] This order is also called the "dictionary order" [see Munkres (2000)] and was known to set theorists and topologists alike; see Sierpinski (1965, p. 221).

[2] In fact, the idea of Wold is so compelling that it is now even included in a basic text on Intermediate Microeconomics to illustrate how a utility function can be found to represent preferences (see Varian (2014)).

We undertake an analysis of the Wold approach with the intention of exploring the role that substitution possibilities play as a sufficient condition for representation. In doing so, our intention is to preserve the spirit of his approach in emphasizing the use of easily verifiable assumptions on preferences, giving a notable priority to applicability of our results over abstract generality. Given that objective, we strive of course to provide the weakest possible assumptions on preferences to guarantee representability.

Our approach to Wold-type representation results involves identifying a nonempty subset of the set of alternatives, such that (a) the preference order is representable on this subset, and (b) the subset has the property that for each alternative, there is some element in this subset which is indifferent to it. This two-step procedure is both necessary and sufficient for representability, and is stated as Lemma 1 (in Section 3). The judicious choice of this subset is central to this approach, and in developing Wold-type representation results, we follow his lead in choosing the "diagonal" of the set of alternatives as the subset of interest.

In applying Lemma 1 to obtain useful representation results, step (a) would then involve showing that the preference order is representable on the diagonal. This step cannot be accomplished without further assumptions on the preference order, and to clarify this important point (among others) we analyze in detail the example of Fishburn (1970), in Example 2 of Section 3. We propose two different assumptions (confined to the diagonal), each of which allows us to accomplish step (a): a weak monotonicity assumption (in Theorem 1), and a weak continuity assumption (in Theorem 2), known as scalar continuity. In both cases, step (a) is completed by showing that each of these assumptions ensures the countable order dense property on the diagonal. In this way, we integrate elements of the approaches of Wold (1943) and Debreu (1954) to representation theorems.

With the choice of the diagonal of the set of alternatives as the relevant subset in applying Lemma 1, step (b) of the lemma can be seen as essentially a substitution condition. One possibility in carrying out this step is to assume it directly, as we do in Theorem 1, where we refer to this substitution condition (justifiably) as the Wold condition. The other is to provide more "primitive" conditions which ensure the Wold condition. One such set of conditions is provided in Theorem 2, where we extend the scalar continuity requirement to the entire space of alternatives, and simultaneously assume that for each alternative, there are diagonal elements in its upper and lower contour sets.

Theorems 1 and 2 are, in our opinion, two very useful representation results, oriented to applications. In applying them, one would check the conditions mentioned in the previous two paragraphs, which we consider to be easily verifiable. In particular, in order to apply the results, one does not have to check the countable order dense property. We relate our representation results to comparable results in the literature in Section 3.2.3, indicating that the conditions we use are indeed very weak. We also note (see remarks (iii) and (iv) following Theorem 2) that, while there is considerable overlap of coverage in the two theorems, there are scenarios which are covered by Theorem 2 but not by Theorem 1, and similarly by Theorem 1 but not by Theorem 2.

Theorem 1 cannot be improved by dropping from its statement either of the conditions used. Weak monotonicity on the diagonal is not sufficient by itself to ensure representability, as is clear from the example of lexicographic preferences. And, the example of Fishburn (1970), analyzed in Example 2 in Section 3.2.2, shows that the Wold condition is also not sufficient by itself to ensure representability.

In Example 1 (in Section 3.2.1) we show that, in the statement of Theorem 2, we cannot drop the (non-emptiness) condition that for each alternative, there are diagonal elements in its upper and lower contour sets; so, scalar continuity by itself cannot ensure

representability. Since the Wold condition always ensures the nonemptiness condition, Example 2 also shows Theorem 2 would not be valid if the scalar continuity condition was dropped from its statement.

Finally, we study the possibility of representation when there are absolutely no substitution possibilities: the indifference set pertaining to each bundle is *degenerate*. We present an example (see Example 3 in Section 4) of a preference order for which the indifference set for every alternative is a singleton but the order is still representable. This example shows that substitution conditions (conditions that deal with properties of indifference classes associated with elements of the domain of preferences) cannot be *necessary* for representation.

To understand the significance of our example let us provide a very rough paraphrase of the intuition for representability in the context of consumer demand theory associated with the extent of substitutability from any given consumption bundle. The problem with the lexicographic preference order is that it has no substitution possibilities at all: each point in the commodity space is indifferent only to itself. This entails that the preference order is extremely sensitive (to changes in any direction of the two-dimensional real space) and the set of real numbers is not large enough to capture this sensitivity. If substitution possibilities are present, so that one has non-degenerate "indifference curves", then "many" points can be assigned the same real number, thereby economizing on the use of real numbers and making it feasible to represent the preference order. Perhaps, the most explicit statement of the above intuition for non-representability appears in Mas-Colell et al. (1995, p. 46), in their informal discussion of the lexicographic preference order. "With this preference ordering, no two distinct bundles are indifferent; indifference sets are singletons. Therefore, we have two dimensions of distinct indifference sets. Yet, each of these indifference sets must be assigned, in an order-preserving way, a different utility number from the one dimensional real line".

However, lest the reader gets carried away by this argument, the authors add a sentence of caution, "In fact, a somewhat subtle argument is actually required to establish this claim rigorously". Our example can also be seen as an elaboration on this sentence of caution. For, just like the lexicographic preference order, our example has (strongly) monotone preferences, with indifference sets that are singletons. And yet the example provides in a sense exactly the opposite scenario to the lexicographic order so far as the representation issue is concerned. The example is not entirely straightforward to construct; it draws on methods which appear in the papers by Lindenbaum (1933) and Sierpinski (1934).

2. Preliminaries

2.1. A sequence space

Let $\mathbb{N}$ be the set of non-negative integers, and $\mathbb{R}$ the set of real numbers. Denote $\mathbb{R}^{\mathbb{N}}$ by Z. For $z, z' \in Z$, we write $z' \geq z$ if $z'_t \geq z_t$ for all $t \in \mathbb{N}$; we write $z' > z$ if $z' \geq z$ and $z' \neq z$; and we write $z' \gg z$ if $z'_t > z_t$ for all $t \in \mathbb{N}$.

Let Y be a nonempty connected set in $\mathbb{R}$ such that $[0, 1] \subset Y$ and $X = Y^m$ where $m \in \mathbb{M} = \mathbb{N} \cup \{\infty\}$; such a space will be called a *sequence space*.[3] The constant m-dimensional vector $(1, 1, 1, \ldots)$ is denoted by e; clearly the vector $e \in X$.

A particular subset of X, the diagonal D, is of special significance in our paper. Define:

$$D = \{x \in X : \text{there exists } \lambda \in Y \text{ such that } x = \lambda e\}$$

2.2. Preference orders and utility functions

We will be concerned with preferences between sequences in the sequence space X, which will also be referred to as *alternatives*. These preferences will be expressed by a binary relation $\succsim$ on $X \times X$.[4] Throughout, we will confine our attention to those binary relations $\succsim$ which are *linear orders*; that is, which are complete (for every $x, y \in X$ either $x \succsim y$ or $y \succsim x$ or both must hold) and transitive (for $x, y, z \in X$ if $x \succsim y$ and $y \succsim z$ hold, then $x \succsim z$ must hold). We will often refer to a linear order $\succsim$ on $X \times X$ as a *preference order*. The asymmetric and symmetric parts of $\succsim$ are denoted as usual by $\succ$ and $\sim$, and are interpreted as strict preference and indifference respectively.

A linear order $\succsim$ on $X \times X$ is called *representable* if there is some $u : X \to \mathbb{R}$ such that for all $x, x' \in X$

$x' \succsim x$ if and only if $u(x') \geq u(x)$.

In this case, we refer to u as a *utility function* that represents the preference order $\succsim$.

3. Representation of preference orders

The complete characterization of a preference order which can be represented by a utility function is given by a joint condition on the sequence space X and the preference order $\succsim$ on $X \times X$. It is called the *countable order-dense property* [see Fishburn (1970), Kreps (1988) and Bridges and Mehta (1995) for a statement and proof of this result]. This property holds if there is a countable subset C of X, such that given any $x, x' \in X$ with $x \succ x'$, there is $z \in C$, such that $x \succsim z \succsim x'$.

In general, this countable-order dense property can be difficult to check directly in applications. Its principal merit is that it is a useful method for developing alternative criteria for representation of preference orders, which are themselves more easily verifiable. An exemplar of such easily verifiable criteria is the Wold type of conditions. The contribution of Wold (1943), generally considered to be the first rigorous solution in the economics literature to the problem of representation of preference orders, historically pre-dates by a decade the introduction by Debreu (1954) of the countable order-dense property as a key concept in this literature. However, for the reason mentioned above, it is useful to view the Wold type of conditions for representation of preference orders through the lens of the countable-order dense property, and we explore this point of view in this paper.

Before discussing the conditions which can be easily identified to be of the "Wold type", it is useful to state a more general condition of which the Wold-type conditions are particular cases. This general condition involves the existence of a nonempty subset S of X such that $\succsim$ is representable on S, and having the property that for each $x \in X$ there is some $s \in S$ such that $x \sim s$.[5]

Lemma 1. *A linear order $\succsim$ on $X \times X$ is representable if and only if it satisfies the following condition: there is a nonempty subset S of X such that (a) $\succsim$ is representable on $S \times S$, and (b) for each $x \in X$ there is some $s \in S$ such that $x \sim s$.*

Proof. (Necessity) Necessity of the condition for representability is obvious, since one can choose $S \equiv X$.

(Sufficiency) Let $v : S \to \mathbb{R}$ be a representation of $\succsim$ on $S \times S$. Given any $x \in X$, define:

$$E(x) = \{s \in S : s \sim x\} \tag{1}$$

and note that the set $E(x)$ is a nonempty subset of S. Define the set:

$$u(x) = \{v(s) : s \in E(x)\} \quad \text{for each } x \in X \tag{2}$$

Note that while the set $E(x)$ may have multiple elements, the set $u(x)$ is a singleton. This follows from transitivity of $\succsim$ and the fact that v represents $\succsim$ on $S \times S$. Thus, u is actually a real-valued function on X.

It remains to check that u represents $\succsim$ on $X \times X$. Let $x, x' \in X$, with $x \sim x'$. Then $E(x) = E(x')$ by transitivity of $\succsim$ on $X \times X$, and so $u(x) = u(x')$ by (2).

Next, let $x, x' \in X$, with $x' \succ x$. We can find $s, s' \in S$, such that $x \sim s$ and $x' \sim s'$. Then, by (2), $u(x) = v(s)$ and $u(x') = v(s')$. Since $x' \succ x$, we have $s' \succ s$ (by transitivity of $\succsim$ on $X \times X$), and so $v(s') > v(s)$, since v represents $\succsim$ on $S \times S$. Thus, $u(x') > u(x)$. ∎

Informally stated, use of the above result involves identifying judiciously a subset S of X, which (a) has some restrictive feature, making the representation on S relatively easier to check than on X itself, and yet is (b) "rich enough" so that it can effectively stand in as a proxy for X, so far as the indifference relation is concerned. Thus, a balancing act is required. Note that, compared with the countable order dense property where a similar judicious choice of a *countable* set is involved, S is very likely to be *uncountable*.

Wold himself chose S to be the diagonal D of X, and in presenting two results of the Wold type (in the subsections below), we will focus on his choice. This choice has two important consequences. It makes the representation issue that one needs to verify in the first part of the condition (in Lemma 1) as one which is essentially confined to the reals. If this more restricted representation issue is going to be addressed by appealing to the countable order dense property (as we hinted at above), then the set of rationals will very likely play an important role.

It also makes us recognize that the second part of the condition (in Lemma 1) is essentially related to the economic idea of substitution, of being compensated for the loss of some portion of one desirable good by more of another desirable good. This role of substitution possibilities, which in our opinion underlies the Wold approach, is difficult to see directly in the characterization of representability by the countable order dense property.

3.1. Wold type representation with monotonicity

Wold (1943) proposed that if preferences were monotone, and if for every consumption bundle, there was a unique diagonal bundle to which it was indifferent, then the scalar associated with that diagonal bundle can be used as a numerical measure of the utility of the consumption bundle. The condition used by Wold is not necessary, even for the class of monotone preferences. Nevertheless, because of its transparent geometric intuition, the method of Wold has been widely used to establish numerical representation of preference orders under a variety of different assumptions on preferences [In the context of intertemporal social preferences, see Diamond (1965), Asheim et al. (2012), Mitra and Ozbek (2013), Banerjee (2014), among others].

In this subsection, we present our version of Wold's representation theorem. In our version, we use significantly weaker assumptions. Thus, we allow for preferences which are not necessarily monotone on the entire space of alternatives; we impose the monotonicity requirement only on the diagonal, and only in its weak form.

Diagonal Monotonicity (DM): If $\lambda, \mu \in Y$, and $\lambda > \mu$, then $\lambda e \succsim \mu e$.

Further, given an alternative, the requirement that there be a *unique* diagonal bundle to which it is indifferent, is also not

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[4] The phrase "a binary relation on X" is often used for a binary relation on $X \times X$.

[5] This kind of two-step approach is not new. Explicit use of this approach is made in Arrow and Hahn (1971), Monteiro (1987), Beardon and Mehta (1994) and Weibull and Voorneveld (2016), among others.

necessary. We impose only the requirement that given an alternative, there *exists some* diagonal bundle to which it is indifferent.[6]

Wold Condition (W): For every $x \in X$, there is $\lambda \in Y$, such that $x \sim \lambda e$.

An important observation about our version of Wold's representation result is that it does not use *any* condition which involves sensitivity of preferences. Thus, the preferences can be entirely insensitive to changes in the alternatives, or insensitive to such changes in parts of the relevant space of alternatives.

Theorem 1. *Let $\succsim$ be a preference order on $X \times X$ which satisfies conditions **DM** and **W**. Then, $\succsim$ is representable.*

Proof. We will use Lemma 1 to establish the result. To this end, choose $S \equiv D$. Then, the second part of the condition used in Lemma 1 is satisfied by using condition **W**. Thus, it remains to verify that $\succsim$ can be represented on D, and this is accomplished by using condition **DM**.

Let us define the set:

$$C = \{\lambda e : \lambda \in Y \text{ is rational}\}$$

Then C is a countable set.

Let $x, x' \in D$, with $x' \succ x$. Then, there exist $\lambda, \lambda' \in Y$ such that $x = \lambda e$ and $x' = \lambda' e$, and:

$$\lambda' e \succ \lambda e \qquad (3)$$

It follows from (3) that $\lambda' \neq \lambda$. If $\lambda > \lambda'$, then by condition **DM**, we would have:

$$\lambda e \succsim \lambda' e$$

But this would contradict (3). Thus, we must have $\lambda \leq \lambda'$, and since $\lambda' \neq \lambda$, we can infer that:

$$\lambda' > \lambda \qquad (4)$$

Using (4), we can choose a rational $\mu \in Y$, such that: $\lambda' > \mu > \lambda$. Using condition **DM**, and $\lambda' > \mu$, we obtain: $\lambda' e \succsim \mu e$. Similarly, using condition **DM**, and $\mu > \lambda$, we obtain: $\mu e \succsim \lambda e$. Thus, we have:

$$x' \equiv \lambda' e \succsim \mu e \succsim \lambda e \equiv x$$

That is, C is a countable order dense subset of D. Applying Lemma II in Debreu (1954, p. 161), the preference order $\succsim$ can be represented on D. ∎

Remarks. (i) It is clear that Theorem 1 is valid when condition **DM** is replaced by: if $\lambda, \mu \in Y$, and $\lambda > \mu$, then $\mu e \succsim \lambda e$. That is, the requirement is that the diagonal monotonicity holds uniformly in one direction or the other. Our theorem uses **DM** because this is the version commonly used in the theory of consumer preferences when the components of an alternative x are desirable goods; it is also natural in the theory of intertemporal social preferences, indicating positive association between individual utilities (components of an alternative x) and social preferences.

(ii) Theorem 1 would not be valid if we dropped the condition **DM** from its statement. This can be seen from Example 2 discussed in Section 3.2.2. It would also not be valid if we dropped the condition **W** from its statement, since the lexicographic preference order in $\mathbb{R}^2$ satisfies **DM** but is not representable.

(iii) Conditions related to (and stronger than) **DM** have been used in the literature on representation of preference orders. Monotonicity is often invoked on the entire space of alternatives, in its weak (Mitra and Ozbek, 2013) or strong forms, as defined below.

Monotonicity (M): For $x, y \in X$ with $x \geq y$ we must have $x \succsim y$.

Strong Monotonicity (SM): $\succsim$ satisfies Monotonicity and when $x > y$, we have $x \succ y$.

The strong form **SM** is referred to as the Pareto condition (or the Strong Pareto condition) in the context of intertemporal social preferences defined on the utility streams of various generations.

On the other hand, a stronger monotonicity condition than **DM**, but confined to the diagonal, has also been used (Banerjee, 2014).

Diagonal Pareto (DP): For $\lambda, \mu \in Y$ with $\lambda > \mu$ we must have $\lambda e \succ \mu e$.

3.2. Wold type representation without monotonicity

The general perception is that some form of monotonicity of preferences is needed for a Wold-type representation result, even though it is recognized that the monotonicity is of a weak form (see, for instance, Beardon and Mehta (1994)). Theorem 1 is definitely of this mold, and one can see how the weak form of monotonicity, imposed only on the diagonal (condition **DM**) suffices to guarantee representation, in the presence of the Wold substitution condition (condition **W**).

However, Lemma 1 indicates that the Wold-type of representation theorem need not be inherently dependent on monotonicity of preferences, even in a weak form. The two-part condition used in Lemma 1 points to the general approach underlying Wold-type representation theorems, and prompts us now to present such a result without monotonicity.

The concept that replaces monotonicity is a weak form of continuity known as *scalar continuity*, which has already been used in representation results by Mitra and Ozbek (2013) and Banerjee (2014).

For each $x \in X$, let us define:

$$A(x) = \{\lambda \in Y : \lambda e \succsim x\} \text{ and } B(x) = \{\lambda \in Y : x \succsim \lambda e\}$$

Then the requirement of scalar continuity is formally expressed as follows.

Scalar Continuity (SC): For each $x \in X$, the sets $A(x)$ and $B(x)$ are closed subsets of Y.

However, we do need to guarantee both parts of the condition in Lemma 1, and we ensure this by explicitly postulating:

Non-Emptiness (NE): For each $x \in X$, the sets $A(x)$ and $B(x)$ are nonempty subsets of Y.

Before proceeding to our next result, let us note that, in our approach to Theorem 1, condition **DM** ensures representation on the diagonal (corresponding to condition (a) in Lemma 1), and condition **W** ensures the substitution feature of condition (b) in Lemma 1. This clean division of the roles of the assumptions in Theorem 1 does not hold in our next result (Theorem 2). Thus, choosing $S = D$ and using Lemma 1, we note that condition **SC** by itself can be used to validate part (a) of the condition in Lemma 1; however, we use both condition **SC** and condition **NE** to ensure part (b) of the condition of Lemma 1.

There is a payoff, though, of this observation. If one is seeking more primitive assumptions which imply condition **W** (in applying Theorem 1), it is reassuring to know that conditions **SC** and **NE** do ensure that condition **W** holds. That is, conditions **SC** and **NE** jointly ensure the substitution possibilities that underlie the Wold approach.

The relationship between condition **W** and condition **NE** (by itself) may also be noted. Clearly **W** implies that **NE** holds. The converse is not true, as is clear from the lexicographic preference order in $\mathbb{R}^2$.

The monotonicity condition **M** also ensures that **NE** is satisfied. But, even though **M** also ensures **DM**, it does not suffice to yield

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representation, as is clear from the lexicographic preference order in $\mathbb{R}^2$. It also illustrates the fact that condition **M** does not imply condition **W**.

Theorem 2. *Let $\succsim$ be a preference order on $X \times X$ which satisfies conditions **SC** and **NE**. Then, $\succsim$ also satisfies condition **W**, and is representable.*

Proof. Given any $x \in X$, we know that $A(x)$ and $B(x)$ are both nonempty (by condition **NE**), and $A(x) \cup B(x) = Y$ (since $\succsim$ is a preference order). Further, by condition **SC**, we know that $A(x)$ and $B(x)$ are closed subsets of Y. Since Y is connected, we infer that: $A(x) \cap B(x) \neq \phi$. This implies that condition **W** holds.

We can use Lemma 1 to establish the representation result. To this end, choose $S \equiv D$. Then, condition (b) in Lemma 1 is satisfied by using condition **W**. Thus, it remains to verify that $\succsim$ can be represented on D.

Let us define the set:

$$C = \{\lambda e : \lambda \in Y \text{ is rational}\}$$

Then C is a countable set.

Consider any $x, y \in D$ with $x \succ y$. The sets $A(x)$ and $B(y)$ are nonempty (since $x, y \in D$) and closed subsets of Y (by condition **SC**). Further, they are disjoint. [For if there is some $\lambda \in A(x) \cap B(y)$, then $\lambda e \succsim x \succ y \succsim \lambda e$, which contradicts the fact that $\succsim$ is a preference order]. Since Y is a connected set, we cannot have $A(x) \cup B(y) = Y$. Thus, there is some $\lambda' \in Y$, such that $\lambda' \notin A(x)$ and $\lambda' \notin B(y)$. That is, $x \succ \lambda' e$ and $\lambda' e \succ y$. Since $\lambda' \in Y$, we can find a sequence $\{\lambda^n\}$ of rationals in Y, such that $\lambda^n \to \lambda'$ as $n \to \infty$.

We claim that there is $N(x) \in \mathbb{N}$, such that for all $n \geq N(x)$, we have $x \succ \lambda^n e$. For, if this does not hold, then there is a subsequence $\{\lambda^{n_r}\}$ of $\{\lambda^n\}$, such that $\lambda^{n_r} \to \lambda'$ as $n_r \to \infty$, and $\lambda^{n_r} e \succsim x$ for all n_r. Since $A(x)$ is a closed subset of Y (by condition **SC**) we must therefore have $\lambda' e \succsim x$, which contradicts the fact that $x \succ \lambda' e$. This establishes the claim. We can similarly establish the claim that there is $N(y) \in \mathbb{N}$, such that for all $n \geq N(y)$, we have $\lambda^n e \succ y$.

Pick any $n \geq \max\{N(x), N(y)\}$. Then, we have $x \succ \lambda^n e \succ y$. Since $\lambda^n e \in C$, we have now found an element $z \in C$ such that $x \succ z \succ y$. That is, the preference order $\succsim$ has the countable order dense property on D and is therefore representable by Lemma II in Debreu (1954). ∎

Remarks. (i) A useful observation that emerges from establishing both Theorems 1 and 2 by using Lemma 1 is that *either* **DM** *or* **SC** will ensure the representability of a preference order $\succsim$ on the diagonal D. Further, it will be noted from the proof of Theorem 2 that in verifying the representability of $\succsim$ on D, one only uses **SC** confined to D. The example of Fishburn (1970), which we analyze in detail in Example 2, shows that one would need *some* condition to ensure the representability of a preference order $\succsim$ on the diagonal D.

(ii) The lexicographic preference order $\succsim$ on $\mathbb{R}^2$ is not representable. It is interesting to view this non-representation result in the light of the two representation theorems presented above (at the risk of repeating some of our earlier observations regarding it). The lexicographic preference order $\succsim$ clearly satisfies **DM**, but Theorem 1 is inapplicable because condition **W** fails. On the other hand, it clearly satisfies **NE**, but Theorem 2 is inapplicable because condition **SC** fails. In terms of our unified treatment relying on Lemma 1, it is worth observing that its inapplicability arises from the failure of condition (b). By choosing $S = D$ in Lemma 1, we can ensure representability of the lexicographic preference order on D by using **DM**. But, this representability cannot be extended to the rest of the space $X = \mathbb{R}^2$ because **W** fails; even though **NE** is satisfied, recall that we used both **SC** and **NE** to ensure **W**, and **SC** fails for the lexicographic preference order.

(iii) Theorem 2 does expand on Theorem 1 in covering cases of Wold-type representation of preference orders. We now present an example of an instance where representation follows, using Theorem 2, even when **DM** fails (so that Theorem 1 is inapplicable).

Consider the following binary relation on $X \equiv Y^2$ where $Y = [0, 1]$. For $x, y \in X$ we define:

$$(x_1, x_2) \succsim (y_1, y_2) \text{ iff } f\left(\frac{x_1 + x_2}{2}\right) \geq f\left(\frac{y_1 + y_2}{2}\right) \tag{5}$$

where $f : [0, 1] \to [0, 1]$ is given by

$$f(t) = t - t^2 \text{ for } t \in [0, 1]$$

Clearly from the definition of $\succsim$ it follows that $\succsim$ is a preference order. To verify **SC**, let $x \in X$ and $\lambda^n \in [0, 1]$ for each $n \in \mathbb{N}$, with $x \succsim (\lambda^n, \lambda^n)$. Under the assumption that λ^n converges to some λ as $n \to \infty$ we need to show that $x \succsim (\lambda, \lambda)$. We obtain from $x \succsim (\lambda^n, \lambda^n)$ and (5) that $f((x_1 + x_2)/2) \geq f(\lambda^n)$ holds. Since f is continuous on $[0, 1]$ and $\lambda \in [0, 1]$, it follows that $f(\lambda^n) \to f(\lambda)$ as $n \to \infty$. Finally as weak inequalities are preserved in the limit we must have $f((x_1 + x_2)/2) \geq f(\lambda)$ implying $x \succsim (\lambda, \lambda)$ (from (5)). This establishes that $B(x)$ is closed. A similar argument also shows that $A(x)$ is closed for every x in X, and establishes **SC**.

Condition **NE** follows easily since it can be verified that $(1/2, 1/2) \in A(x)$ and $(0, 0) \in B(x)$ for all $x \in X$. Theorem 1 (or direct inspection of the definition itself) can be invoked to claim that the order is representable. In fact, the stronger condition **W** is also satisfied by $\succsim$. To see this, note that given any $x \in X$, we can define $\lambda = (x_1 + x_2)/2$. Then, $\lambda \in [0, 1]$, and we have: $f((x_1 + x_2)/2) = f((\lambda + \lambda)/2)$, so $\lambda e \sim x$.

The nature of the function f guarantees that the preference order $\succsim$ violates condition **DM**. This follows from noting that f attains its (unique) maximum at $(1/2)$ and is strictly increasing in the sub-domain $[0, 1/2)$ and strictly decreasing in the sub-domain $(1/2, 1]$.

(iv) On the other hand, Theorem 1 is applicable in some cases, where Theorem 2 is not. We now present an example of an instance where representation follows, using Theorem 1, even when **SC** fails (so that Theorem 2 is inapplicable).

Let $X = Y^2$, where $Y = [0, 1]$. Define $u : X \to \mathbb{R}$ by:

$$u(x_1, x_2) = \begin{cases} 0 & \text{for } (x_1 + x_2)/2 \leq (1/2) \\ 1 & \text{for } (x_1 + x_2)/2 > (1/2) \end{cases} \tag{6}$$

Now, define the binary relation $\succsim$ on $X \times X$ by:

$$x' \succsim x \text{ if and only if } u(x'_1, x'_2) \geq u(x_1, x_2) \tag{7}$$

Clearly from (6) and (7) it follows that $\succsim$ is a preference order. It satisfies condition **DM**, since if λ and λ' belong to $[0, 1]$, and $\lambda' > \lambda$, then $u(\lambda' e) = 0$ and $u(\lambda e) = 0$ when $\lambda' \leq (1/2)$, and $u(\lambda' e) = 1$ while $u(\lambda e) \leq 1$ when $\lambda' > (1/2)$. Thus, in either case $\lambda' e \succsim \lambda e$.

Given any $x \in X$, we can define $\lambda = (x_1 + x_2)/2$. Then, $\lambda \in [0, 1]$, and for $\lambda e \in D$, we have:

$$(\lambda + \lambda)/2 = \{[(x_1 + x_2)/2] + [(x_1 + x_2)/2]\}/2 = (x_1 + x_2)/2$$

and so $u(x_1, x_2) = u(\lambda e)$. Thus, $\lambda e \sim x$, and so condition **W** is satisfied.

We can now use Theorem 1 to claim that $\succsim$ has a representation. Of course, we already know that u given by (6) is a representation of $\succsim$; but the point to be made here is that Theorem 1 is applicable to the preference relation defined by (7) to guarantee a representation.

Theorem 2 is, however, not applicable, because condition **SC** is violated. To see this, define $x = (2/3)e$, and note that $u(x) = 1$. Now, $A(x) = \{\lambda \in [0, 1] : \lambda e \succsim x\} = (\frac{1}{2}, 1]$ which is not a closed subset of Y.

3.2.1. Role of condition NE

Condition **NE** used in Theorem 2 may be viewed as a condition indicating compensation possibilities but (as already noted above) it is weaker than the substitution condition **W** of Wold. [Condition **W** implies that **NE** holds; the converse is not true, as is clear from the lexicographic preference order in $\mathbb{R}^2$].

In this section we study its role in Theorem 2 by providing an example of a preference order which satisfies **SC** but **NE** *fails* and the order is *not* representable. Thus Theorem 2 would not be valid if we drop **NE** from its statement.

Example 1. Consider the sequence space $X = Y^2$ where $Y = [0, 1]$. Define the following subsets of X : $V = \{x \in X : x_1 = 1 \text{ and } 0 \le x_2 < 1\}$, $D = \{x \in X : \text{there is } \lambda \in Y \text{ such that } x = (\lambda, \lambda)\}$ and $R = X \setminus (V \cup D)$.

Define a binary relation $\succsim$ on $X \times X$ as follows: (i) For any $x, y \in V \cup R$ we say $x \succsim y$ iff $x \ge_L y$ (where $\ge_L$ is the standard lexicographic ordering on $\mathbb{R}^2$); (ii) for any $x \in D$ and $y \in R$ we define $x \succ y$, (iii) for $[x \in V \text{ and } y \in D]$ or $[x \in D \text{ and } y \in V]$ or $[x, y \in D]$ we declare $x \succsim y$ iff $\min\{x_1, x_2\} \ge \min\{y_1, y_2\}$.

Preference order: Completeness follows from observing that the sets V, D and R on which $\succsim$ is defined along with (i)–(iii) exhausts all possibilities of comparisons for ordered pairs in X.

Transitivity: Let $x, y, z \in X$ such that $x \succsim y$ and $y \succsim z$. We need to show that $x \succsim z$ holds. Transitivity is verified for the binary relation by considering different cases pertaining to membership of pairs across the definitions (i)–(iii).

Before coming to the proof of transitivity, let us make a preliminary point. If $x, y \in X$ are such that both belong to V, then since $x_1 = 1 = y_1$, (i) tells us that $x \succsim y$ iff $x_2 \ge y_2$. In other words, $x \succsim y$ iff $\min\{x_1, x_2\} \ge \min\{y_1, y_2\}$. This leads us to the following observation, using (iii):

For $x, y \in V \cup D$, we have $x \succsim y$ iff $\min\{x_1, x_2\} \ge \min\{y_1, y_2\}$ (O)

We now come to the proof of transitivity, and we break up our analysis into two cases: (I) $x \in R$; (II) $x \in V \cup D$.

Case (I): In this case $x \succsim y$ dictates (by (i) and (ii)) that $y \in R$ and $y \succsim z$ similarly dictates that $z \in R$. Thus, we have $x, y, z \in R$ and transitivity of $\succsim$ follows from the transitivity of the lexicographic order.

Case (II): The following exhaustive possibilities need to be addressed: (a) $y \in R$ and $z \in V \cup D$, (b) $y, z \in R$, (c) $y \in V \cup D$ and $z \in R$, (d) $y \in V \cup D$ and $z \in V \cup D$.

Observe that from the definition of the binary relation $\succsim$ it follows that if $y \in R$ and $y \succsim z$, then $z \notin V \cup D$ (by (i) and (ii)). This means possibility (a) cannot occur. In possibilities (b) and (c), we have $z \in R$, while $x \in V \cup D$. Thus, by (i) and (ii), we have $x \succ z$, establishing transitivity for these possibilities. So, we are left only with the possibility (d), where we have $x, y, z \in V \cup D$. Here, $x \succsim y$ dictates (by (O)) that $\min\{x_1, x_2\} \ge \min\{y_1, y_2\}$ and $y \succsim z$ similarly dictates (by (O)) that $\min\{y_1, y_2\} \ge \min\{z_1, z_2\}$. Thus, we have $\min\{x_1, x_2\} \ge \min\{z_1, z_2\}$. Since $x, z \in V \cup D$, (O) now dictates that $x \succsim z$, establishing transitivity for possibility (d).

Condition SC: Consider $x \in R$. Observe that by (ii) for any $y \in D$ we have $y \succ x$, showing that for each $x \in R$ the set $A(x)$ must be $[0, 1]$, hence closed. For $x \in D$ (with $x = (\lambda, \lambda)$) we must have $A(x) = [\lambda, 1]$ (using (iii)) when $\lambda < 1$ and $A(x) = \{1\}$ with $\lambda = 1$; in both cases $A(x)$ is closed. For every $x \in V$, using (iii) in the definition of $\succsim$ we must have $A(x) = [x_2, 1]$. This shows that $A(x)$ is closed when $x \in V$ as well.

Now consider $B(x)$ for $x \in R$. Observe that by (ii) for any $y \in D$ we have $y \succ x$, showing that $B(x)$ is empty, hence closed. For $x = (\lambda, \lambda) \in D$ we obtain $B(x) = [0, \lambda]$ whenever $\lambda > 0$ and

$B(x) = \{0\}$ when $\lambda = 0$ showing that $B(x)$ is closed in Y. When $(x_1, x_2) \in V$ we get $B(x) = [0, x_2]$ when $x_2 > 0$ and $B(x) = \{0\}$ when $x_2 = 0$ (using (iii)), showing that $B(x)$ is closed when $x \in V$ as well.

Condition NE: Observe that $B(x)$ is empty for every $x \in R$, so **NE** fails to hold.

Representation: Suppose $u : X \to \mathbb{R}$ represents $\succsim$. Define the subset:

$W = \{(x_1, x_2) \in X : 0 < x_1 < 1 \text{ and } x_2 \in \{1, 0\}\}$

The set W is a subset of R, since W has empty intersection with D and V.

For each $\alpha \in (0, 1)$ we observe that $(\alpha, 1), (\alpha, 0) \in W \subset R$, and using (i), we have $(\alpha, 1) \succ (\alpha, 0)$. Thus, we obtain $u((\alpha, 1)) > u((\alpha, 0))$. Denote the interval $(u((\alpha, 0)), u((\alpha, 1)))$ by $I(\alpha)$. Now, let α, β be arbitrary elements of $(0, 1)$ with $\beta > \alpha$. Using (i), we have $(\beta, 0) \succ (\alpha, 1)$ so that $u((\beta, 0)) > u((\alpha, 1))$. Thus, the interval $I(\beta)$ lies entirely to the right of $I(\alpha)$ on the real line. Since $(0, 1)$ is uncountable, we have an uncountable collection of non-overlapping open intervals on the real line. This leads us to a contradiction, since each of these intervals must contain a distinct rational, and the set of rationals is countable.

3.2.2. Role of condition SC

As already mentioned (in remark (ii) following Theorem 2), the lexicographic preference order in $\mathbb{R}^2$ shows that Theorem 2 would not be valid if condition **SC** is omitted from its statement.

We know that conditions **SC** and **NE** jointly imply the substitution condition **W** of Wold (see Theorem 2), which by itself implies condition **NE**. So, a legitimate question is whether the substitution condition **W** of Wold, by itself, is sufficient for representability of a preference order. The following example, due to Fishburn (1970, p. 27), shows that this is not the case, and therefore limits the role of substitutability in representation results.

Fishburn's example is chosen in this context as the relevant preference order is defined on an interval of the reals, so that condition **W** (and therefore condition **NE**) is trivially satisfied.

Example 2. Let $Y = [-1, 1]$ and $X = Y$. Define the binary relation $\succsim$ as follows. For $x, y \in X$,

$x \succsim y$ iff $(|x|, x) \ge_L (|y|, y)$

where $\ge_L$ is the standard lexicographic order in $\mathbb{R}^2$.

It is straightforward to check that the binary relation $\succsim$ is complete and transitive, and so is a preference order. Further, defining as usual $\succ$ and $\sim$ as the asymmetric and symmetric components of $\succsim$, we note that, for $x, y \in X$:

$x \succ y$ iff either (i) $|x| > |y|$, or (ii) $(|x| = |y|, \text{ and } x > y)$ (8)

and for $x, y \in X$:

$x \sim y$ iff $x = y$ (9)

Representation: Suppose that $\succsim$ can be represented by a utility function $u : X \to \mathbb{R}$.

Define $Z = (0, 1]$, and note that for each $x \in Z$, we have $|-x| = |x|$, and $-x < 0 < x$, so that $x \succ -x$ by (8). Thus, we must have:

$u(x) > u(-x)$ (10)

Define $I(x) = [u(-x), u(x)]$ for each $x \in Z$. Then, by (10), $I(x)$ is a closed, non-degenerate interval in $\mathbb{R}$.

Now, consider arbitrary $x, y \in Z$, with $y > x$. Then, $I(x) = [u(-x), u(x)]$ and $I(y) = [u(-y), u(y)]$. Note that $|-y| = y > x = |x|$, and so by (8), we must have $(-y) \succ x$. Since u represents $\succsim$,

we then obtain:

$$u(-y) > u(x)$$

Thus, $I(y)$ is a closed degenerate interval of $\mathbb{R}$, which lies entirely to the right of $I(x)$ on the real line. Since Z is uncountable, we have an uncountable collection of non-overlapping non-degenerate closed intervals on the real line. This is a contradiction, since each of these intervals must contain a distinct rational, and the set of rationals is countable.

Condition SC: For $x \in (0, 1)$, we have $A(x) = [-1, -x] \cup [x, 1]$, which is not closed, so condition **SC** is violated.

Condition W: For every $x \in X$, we have $xe \equiv x \sim x$, so **W** is trivially satisfied. In fact, for each $x \in X$, the set $I(x) = A(x) \cap B(x)$ is a *singleton* (by (9)) so even this more stringent requirement (that is, $I(x)$ being a singleton) does not guarantee representability.

3.2.3. Related literature

In this subsection we make a few remarks relating our representation result in Theorem 2 (and in Theorem 1) to some results already available in the literature.

(i) Perhaps, the results most directly comparable to Theorems 1 and 2 are contained in Mitra and Ozbek (2013) and Banerjee (2014), which unify many of the representation results appearing earlier in the intertemporal social literature.

In the former, Conditions **SC** and **M** are combined to obtain a representation result, which is then seen to unify several representation results appearing in the literature on intertemporal social choice. This can be seen to be a special case of Theorem 2, since **M** implies that condition **NE** holds. It can also be seen to be a special case of Theorem 1, since **M** implies both **DM** and **NE**, while **SC** combined with **NE** ensures **W**.

Banerjee (2014) maintains **SC**, but replaces **M** by conditions **DP** and **NE**. His result is a special case of Theorem 2, and in fact shows that for his result, condition **DP** is redundant. His result also follows from Theorem 1, since **DP** implies **DM**, while **SC** combined with **NE** ensures **W**.

(ii) Monteiro (1987) considers a connected, separable subset F of X, having the property that for every $x \in X$ there is some $a, b \in F$ such that $a \succsim x \succsim b$, called an order boundedness property. This is used along with closed upper and lower contour sets (known as order continuity) to establish a representation result. Condition **NE** can be seen to be an order boundedness property when F is chosen to be D. Our preference for using **NE** may be seen to be in keeping with our motivation of emphasizing ease of application over generality.

It is worth pointing out that when $X = Y^{\mathbb{N}}$, order continuity (as assumed in Monteiro (1987)) would be equivalent to continuity defined using the sup-norm topology. Scalar continuity is known to be weaker than sup-norm continuity in this context (see Mitra and Ozbek (2013)). This shows that Theorem 2 does not directly follow from that of Monteiro (1987).

Theorem 2 does generalize to well-behaved metric spaces that have the sequence space structure. One such generalization is provided in the (Appendix A.2). As is evident from the proof of Theorem 2, the diagonal of the sequence space facilitates the intuitive argument at the core of the Wold method; however, the argument itself is not restricted by the role of the specific diagonal set. This generalization is verified in Banerjee and Mitra (2018).

(iii) The idea that representability along the diagonal can be extended to the representability of the order itself can be viewed as measuring changes in utility across points on the diagonal as a proxy for the difference in utilities across two bundles (to which the diagonal elements are indifferent). With this view in mind, the Wold approach to utility representation is applicable to a variety of measurement problems giving rise to "path-based measures".

A (non-exhaustive) list of papers that develop the Wold idea in the context of abstract measurement problems include Debreu (1951), Farrell (1957), Hougaard and Keiding (1998) and Chambers and Miller (2014) (in the context of efficiency measurement), and Kalai (1977) and Thomson and Myerson (1980) (for solutions to bargaining problems) among others.

4. Degenerate indifference sets and representation

In Section 3, we have presented Wold-type representation results, where existence of substitution possibilities plays a key role. We have also seen that the presence of substitution possibilities alone need not guarantee representation of preference orders.

In this section, we make an entirely different point about the role of substitution possibilities in representation results. It is that representation of preference orders is possible even when there is a complete absence of substitution possibilities. We provide an example of a *strongly monotone* preference order whose indifference classes are *singleton* sets, which is nevertheless *representable*.

This example runs counter to the intuition that non-degenerate indifference classes (sets having more than one point) are *necessary* for representation. Actually, much of this intuition is based on our familiarity with the lexicographic preference order, which is strongly monotone with singleton indifference sets, and is *not representable*. The example provides in a sense exactly the opposite scenario to the lexicographic order so far as the representation issue is concerned, and is instructive in correcting that intuition.

In particular, the example shows that the substitution condition **W** of Wold is not necessary for representation, even for the class of preference orders satisfying the monotonicity condition **M**.

Example 3. Consider the sequence space $X = Y^2$ with $Y = [0, 1]$. For any $p \in Y$ we can express p in its binary expansion form as follows:

$$p = \frac{a_1}{2} + \frac{a_2}{2^2} + \frac{a_3}{2^3} + \cdots + \frac{a_n}{2^n} + \cdots \tag{11}$$

where $a_i \in \{0, 1\}$ for each $i \in \mathbb{N}$. It is well known that every real in the interval $[0, 1]$ can be expressed in the form (11) (also written as $a_1 a_2 \ldots$) and when such a representation is not unique there are precisely two such representations (for instance $01111\ldots$ and $1000\ldots$ both represent $(1/2)$).[7] The analysis can now proceed by stating that to make the binary expansion unique we will use the terminating expansion that ends with all $0's$ in the event of non-uniqueness. Keeping this convention in mind, we will say that p has the *binary representation* $\{a_i\}$ when p can be expressed as (11) using the sequence $\{a_i\}$.

Now define $f : Y \to Y$ by

$$f(p) = \frac{a_1}{4} + \frac{a_2}{4^2} + \frac{a_3}{4^3} + \cdots + \frac{a_n}{4^n} + \cdots \tag{12}$$

and $h : Y \to Y$ by

$$h(q) = f(q)/2 \tag{13}$$

Finally define $u : X \to \mathbb{R}$ by

$$u(x_1, x_2) = f(x_1) + h(x_2) \quad \text{for all } (x_1, x_2) \in X \tag{14}$$

Using u, define the binary relation $\succsim$ on $X \times X$ as:

For all $x, y \in X$, $\quad x \succsim y$ iff $u(x_1, x_2) \geq u(y_1, y_2)$ (15)

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[7] An equivalent way of stating this, following Royden (1988, p. 40) would be to say that expression (11) is non-unique only in the case where p is of the form $q/2^n$ where $0 < q < 2^n$ is an integer.

Preference order: The definition of $\succsim$ in (15) is made using a real-valued function u as defined in (14), which immediately implies that $\succsim$ is complete and transitive.

Strong monotonicity: Since f and h are strictly increasing on Y strong monotonicity of $\succsim$ on X follows immediately from (14) and (15).

Before proceeding further, we make the following observation (the proof of which is included in an Appendix).

Observation. *Let* $1 \geq p' > p \geq 0$ *and* $\{a_i\}$ *and* $\{a_i'\}$ *be the standard binary expansions of* p *and* p' *respectively. Denote* $m(p, p') \equiv \min\{i \in \mathbb{N} : a_i \neq a_i'\}$ *by* r. *Then the following three properties hold: (a)* $a_r' = 1$, $a_r = 0$, *(b) there is some* $n \geq (r+1)$ *such that* $(a_n' - a_n) \in \{0, 1\}$, *(c)* $(4/3)/4^r \geq [f(p') - f(p)] > (2/3)/4^r$.

Degenerate indifference sets: We now turn to the demonstration that all indifference sets of $\succsim$ are degenerate. Suppose on the contrary, there is x, x' in X such that $x \neq x'$ and $u(x) = u(x')$. For convenience, write $x = (p, q)$ and $x' = (p', q')$. As g, h are strictly increasing functions on Y, we can assume without loss of generality that $p' > p$ and $q' < q$.

The fact that $u(x') = u(x)$ must yield, applying (14),

$$f(p') - f(p) = h(q) - h(q') \tag{16}$$

Using (13) and (16) we obtain:

$$f(p') - f(p) = (1/2)[f(q) - f(q')] \tag{17}$$

Denote $m(p, p')$ by r and $m(q, q')$ by s. Two exhaustive possibilities emerge: (i) $s \geq r$; (ii) $s < r$. In case (i) using (c) in the Observation above,

$$(1/2)[f(q) - f(q')] \leq \frac{(1/2)(4/3)}{4^s} = \frac{(2/3)}{4^s} \leq \frac{(2/3)}{4^r} \tag{18}$$

Also, using (c) in the Observation above, we get:

$$f(p') - f(p) > \frac{(2/3)}{4^r} \tag{19}$$

Now (18) and (19) contradict (17).

In case (ii) we have $r \geq s+1$ and so, using (c) in the Observation above, we get:

$$f(p') - f(p) \leq \frac{(4/3)}{4^r} \leq \frac{(4/3)}{4^{s+1}} = \frac{(1/3)}{4^s} \tag{20}$$

On the other hand, using (c) in the Observation above, we also have:

$$(1/2)[f(q) - f(q')] > \frac{(1/2)(2/3)}{4^s} = \frac{(1/3)}{4^s} \tag{21}$$

Clearly, (20) and (21) contradict (17).

Thus, every indifference set associated with $\succsim$ must be a singleton or equivalently, degenerate. However, $\succsim$ is representable, which follows directly from (15).

4.0.4. Discussion

(i) In Chipman (1960), Theorems 3.3 and 3.4) a characterization of representable preference orders is presented using an Axiom of Substitution (Axiom 4, p. 214). Our Example 3 shows that there is a preference order where there are no substitution possibilities, yet the preference order is representable. Hence, it is natural to ask how Chipman's result relates to our Example 3. Informally, we argue that within the framework of Example 3 the non-representable preferences that Chipman characterizes are necessarily lexicographic. The Axiom of Substitution that he uses to characterize representability captures more than what is intuitively understood by substitution and hence, our Example 3 does not refute Chipman's characterization.

To keep the discussion self-contained and free of detailed topological concepts, we concentrate on preference orders $\succsim$ defined on $X = Y^2$, where $Y = [0, 1]$. Let $\succsim$ be a strongly monotone preference order. We will say that an ordered pair $(X, \succsim)$ can be *embedded* in $([0, 1] \times [0, 1], \geq_L)$ where $\geq_L$ is the standard lexicographic ordering if there is a function $f : X \to X$ such that $x \succsim y$ iff $f(x) \geq_L f(y)$; the ordered pair $(X, \succsim)$ is said to have a *lexicographic embedding* in this case. Notice that order embedding is a generalization of the concept of representability. If the image space of the embedding function f is $\mathbb{R}$, then we have a (real valued) representation.

Our claim is that any ordered pair $(X, \succsim)$ that is strongly monotone, has singleton indifference sets and is not representable can be embedded in $([0, 1] \times [0, 1], \geq_L)$. This follows from Beardon et al. (2002b)[8] on noting that the diagonal D is a completely ordered subset of X and strong monotonicity can be used to show that conditions of their Theorem 2.2 (and Remark 2.1 following it) are satisfied. Therefore, any non-representable, strongly monotone preference order on X with singleton indifference sets, ranks elements of X in accordance to some (non-representable) lexicographic ordering of elements of X. So, for this class of preferences, Chipman's representation results (Theorem 3.3 and Theorem 3.4) characterize those orders that have a lexicographic embedding.

(ii) While the subject of enquiry in Mandler (2017) is different from ours, he provides an example of a Cantor preference ordering that is strongly monotone, has singleton indifference sets and is representable, thereby, meeting all the properties exhibited by Example 3. A similar example was also communicated to us by Ray (2013) in a private communication.

(iii) Banerjee (1964, p. 160–161)) provides an exposition of how existence of non-degenerate indifference sets is ensured if one postulates that preferences are representable by a *continuous* utility function.[9] Note that while the preference order defined in Example 3 is representable, even when the underlying binary relation exhibits a serious dearth of substitution possibilities, such a representation cannot be continuous.

This is an implication of the following mathematical result, stated and proved in Sierpinski (1965, p. 70–71).

There exists no continuous function $f(x, y)$ *of two real variables (even continuous only with respect to each variable separately) on* $X = Y^2$ *with* $Y = [0, 1]$ *which for different pairs of real numbers* (x, y) *would always assume different values.*

5. Conclusion

This paper has produced results on two aspects of representation of preference orders. The first set of results (1, 2) distills the Wold approach, by emphasizing easily verifiable conditions which are sufficient to guarantee representation. We provide several examples to show that our results cannot be improved upon by dropping any of these conditions. The conditions used also illustrate the role that substitution possibilities play in ensuring Wold-type representation results.

Our second contribution addresses the issue of the necessity of substitution possibilities for an order to be representable. We show, by means of an example, that a monotone order can be constructed with no substitution possibilities (each indifference class is a singleton), which is nevertheless representable. Our example provides a clear limit to the substitution implications of representability.

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[8] This paper builds on the comprehensive analysis of non-representability presented in Beardon et al. (2002a).

[9] Actually, Banerjee (1964, p. 161) goes on to say, "Thus iso-utility points (i.e., indifference classes) necessarily exist if we start from a real-valued utility function." But, clearly, he means continuous real-valued utility function, in view of the argument he has provided.

Appendix

A.1. Proof of observation in Example 3

We present in this section the proof of the **Observation** from Example 3 in Section 4.

Proof of Observation. Let a and a' denote the sequences $\{a_i\}$ and $\{a'_i\}$, the binary representations of p, p' respectively. Since $p \neq p'$ there is some i for which $a_i \neq a'_i$ holds, which guarantees that $m(p, p')$ is well defined. There are two possibilities: (i) $a'_r = 0$ and $a_r = 1$; (ii) $a'_r = 1$ and $a_r = 0$.

Suppose (i) is true then,

$$p' - p = \sum_{n=r}^{\infty} \frac{(a'_n - a_n)}{2^n} = -\frac{1}{2^r} + \sum_{n=r+1}^{\infty} \frac{(a'_n - a_n)}{2^n}$$

$$\leq -\frac{1}{2^r} + \sum_{n=r+1}^{\infty} \frac{1}{2^n} = 0$$

which contradicts the fact that we are given $p' > p$. Thus, (i) cannot be true, and (ii) must hold, verifying (a). For ready reference let us note part (a) explicitly as:

If $1 \geq p' > p \geq 0$, then $a'_r = 1$ and $a_r = 0$,

where $r = m(p, p')$ (A.1)

Now consider the possibility that $(a'_n - a_n) = -1$ for all $n \geq r+1$, then

$$p' - p = \sum_{n=r}^{\infty} \frac{(a'_n - a_n)}{2^n} = \frac{1}{2^r} + \sum_{n=r+1}^{\infty} \frac{(a'_n - a_n)}{2^n}$$

$$= -\frac{1}{2^r} - \sum_{n=r+1}^{\infty} \frac{1}{2^n} = 0$$

would again contradict the fact that we are given $p' > p$. This shows that (b) must also be true.

To show (c) evaluate the difference $[f(p') - f(p)]$ as follows:

$$[f(p') - f(p)] = \frac{(a'_r - a_r)}{4^r} + \sum_{n=r+1}^{\infty} \frac{(a'_n - a_n)}{4^n}$$

$$= \frac{1}{4^r} + \sum_{n=r+1}^{\infty} \frac{(a'_n - a_n)}{4^n}$$

$$> \frac{1}{4^r} + \sum_{n=r+1}^{\infty} \frac{(-1)}{4^n}$$

$$= \frac{1}{4^r} - \frac{1}{4^{r+1}} \frac{4}{3} = \frac{1}{4^r} \frac{2}{3} > 0 \quad\quad (A.2)$$

The second line of (A2) follows from the first line using (A1) and strict inequality in the third line follows from noting that $(a'_n - a_n) > -1$ for at least some $n \geq r + 1$. This shows that $[f(p') - f(p)] > \frac{(2/3)}{4^r}$.

From the second line of (A2) using $(a'_n - a_n) \leq 1$ for all $n \geq r+1$ we obtain:

$$[f(p') - f(p)] = \frac{1}{4^r} + \sum_{n=r+1}^{\infty} \frac{(a'_n - a_n)}{4^n}$$

$$\leq \frac{1}{4^r} + \sum_{n=r+1}^{\infty} \frac{1}{4^n} = \frac{1}{4^r} \frac{4}{3} \quad\quad (A.3)$$

proving (c).

A.2. Generalization of Theorem 2

In this subsection we present a generalization of Theorem 2 in Section 3.2, as well as an example to illustrate how it can be used.

Consider a metric space (Y, d). Assume that Y is a connected, separable metric space. A *sequence space* is a Cartesian product of the set Y taken M times, where, $M \in \mathbb{N} \cup \{\infty\}$. Denote the sequence space Y^M by X. Endow X with the sup-norm topology, with $\|x - y\| = \sup\{d(x_i, y_i) : i = 1, \ldots, M\}$.

We will present a result on the representability of preference orders on X along the lines of Wold (1943). Recall that a *homeomorphism* between two topological spaces T and T' is a continuous bijection $f : T \to T'$ such that f^{-1} is also continuous. Define the upper and lower contour sets corresponding to each x in X by:

$$U(x) = \{x' \in X : x' \succsim x\} \; ; \; L(x) = \{x' \in X : x \succsim x'\}$$

Suppose $\succsim$ is a preference order on X. Assume that there is a subset T of X that is homeomorphic to Y^n for some $n \in \mathbb{N}$. Write $A(x)$ for $U(x) \cap T$ and $B(x)$ for $L(x) \cap T$. The following two properties will be used in our representation result below.

Property 1. *For each $x \in X$ the sets $A(x)$ and $B(x)$ are closed in T.*

Property 2. *For each $x \in X$ the sets $A(x)$ and $B(x)$ are nonempty.*

Theorem 3. *If $\succsim$ is a preference order on $X \times X$ that satisfies Properties 1 and 2 with respect to a subset T of X, with T homeomorphic to Y^n for some finite n, then $\succsim$ is representable.*

Proof. Step 1: (For each $x \in X$ there is some $t \in T$ such that $t \sim x$) Since T is homeomorphic to Y^n for some finite n there is a continuous bijection $f : Y^n \to T$ such that f^{-1} is continuous as well. Since Y^n is connected (finite Cartesian product of a connected space is connected, Theorem 23.6, p. 150, Munkres (2000)) and f is a continuous bijection, $f(Y^n) = T$ is also connected (Theorem 23.5, p. 150, Munkres (2000)). As T is connected, and $A(x)$ and $B(x)$ are closed nonempty subsets of the connected space T there must exist some $t \in A(x) \cap B(x)$. This shows that there is some $t \in T$ such that $t \sim x$.

Step 2: (Existence of a countable order dense set of X) Since Y is separable, the space Y^n (endowed with the sup-metric) is also separable. So there is some countable order dense subset Z of Y^n. Write Z as $\{z_1, z_2, \ldots\}$ and Z' as $\{z'_1, z'_2, \ldots\}$ where, $z'_i = f(z_i)$. This makes Z' a countable subset of T. We will show that Z' is a countable order-dense subset of X. We first show that Z' is dense in T. If U is open in T, then $f^{-1}(U)$ is open in Y^n (as f is a homeomorphism). As Z is dense in Y, $f^{-1}(U) \cap Z$ is nonempty. It follows now that $f(z) \in U \cap Z'$.

Suppose $x, y \in X$ and $x \succ y$. By Step 1, there exists $t, t' \in T$ such that $t \sim x$ and $t' \sim y$. Transitivity implies (using $x \succ y$) that $t \succ t'$. Consider the sets $\alpha(t') = \{x \in T : x \succ t'\}$ and $\beta(t) = \{x \in T : t \succ x\}$. By Property 1 both sets are open in T and nonempty (since $t \in \alpha(t')$ and $t' \in \beta(t)$). Since T is connected (as a continuous image of a connected space Y), $\alpha(t') \cap \beta(t)$ must be nonempty [if $\alpha(t') \cap \beta(t)$ is empty, then the open sets $\alpha(t')$ and $\beta(t)$ would constitute a separation of T, contradicting its connectedness]. As both $\alpha(t')$ and $\beta(t)$ are open, $\alpha(t') \cap \beta(t)$ is also open in T. As Z' is dense in T we must have $[\alpha(t') \cap \beta(t)] \cap Z'$ is nonempty. This implies there is some $z' \in Z'$ such that $t \succ z' \succ t'$. Noting that $t \sim x$ and $t' \sim y$ hold we have indeed shown that there is some $z' \in Z'$ such that $x \succ z' \succ y$ establishing that Z' is countable order-dense in X. By Lemma II in Debreu (1954), we conclude that $\succsim$ is representable.

Remark. In order to see how Theorem 2 in Section 3.2 follows from Theorem 3, let I be some non-degenerate interval in $\mathbb{R}$ and

$X = I^M$ for $M \in \mathbb{N} \cup \{\infty\}$. Observe that here $Y = I$ is separable and connected. Let us define $T = \{\lambda e : \lambda \in I\}$. It is easily seen that T is homeomorphic to $Y = I$ with $A(x) = U(x) \cap T$ and $B(x) = L(x) \cap T$. If $\succsim$ is a preference order satisfying conditions **SC** and **NE**, then Properties 1 and 2 are satisfied, and Theorem 3 implies that $\succsim$ is representable.

Example 4 (*Galperti and Strulovici, 2017*)**.** Let Y be a connected, separable metric space and $X = Y^\infty$ (hence, $M = \infty$). Galperti and Strulovici (2017) consider the representability of preference orders on X satisfying the following two conditions:

Continuity (C): For all $c \in X$ the sets $U(c)$ and $L(c)$ are closed in X.

Future Constant-Flow Dominance (FCFD): For all $c \in X$, there exists $x, y \in Y$ such that $(c_1, x, x, \ldots) \precsim c \precsim (c_1, y, y, \ldots)$.

To use Theorem 3, we first define a suitable $T \subset X$ as follows:

$$T = \{x \in X : x = (a, b, b, \ldots) \text{ for some } a, b \in Y\}$$

We now make the following observations:

(i) The map $f : Y^2 \to T$ given by $f(a, b) = (a, b, b, \ldots)$ is a continuous bijection with a continuous inverse. Hence T is homeomorphic to Y^2.

(ii) As $U(c)$ and $L(c)$ are closed in X, the sets $A(c) = U(c) \cap T$ and $B(c) = L(c) \cap T$ are closed in the relative topology on T. So, Property 1 is satisfied.

(iii) Consider the sets $A(c)$ and $B(c)$ defined using this T. We have $A(c) = \{t \in T : t \succsim c\}$ and $B(c) = \{t \in T : c \succsim t\}$. Given any $c \in X$, by FCFD, there is some $x, y \in Y$ such that $(c_1, x, x, \ldots) \precsim c \precsim (c_1, y, y, \ldots)$. Notice that $(c_1, x, x, \ldots), (c_1, y, y, \ldots) \in T$ and hence, FCFD implies that $A(c)$ and $B(c)$ are nonempty, showing that Property 2 is satisfied.

Thus, by Theorem 3, the preference order satisfying C and FCFD is representable.

Remark. While using a stronger assumption on continuity (C) than that stated in Property 1, Galperti and Strulovici (2017) obtain the stronger result of *continuous* representation of altruistic preferences. The representation (not necessarily continuous) of such preferences is a direct consequence of Theorem 3.

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